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NEW PROBABILISTIC STATISTICAL AND DYNAMIC MODELS TO CONTROL LIFE CYCLE IN LACTING COWS

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Abstract

Reproduction in dairy herds recently has become increasingly important. The tendency to its reduction occurs everywhere in all countries with a developed dairy husbandry. On average, the number of lactations per cow is diminishing and now close to 3 while a genetic potential of many cattle breeds is over 10 lactations. To resolve this issue, a theoretical base should be developed using latest progress in various related sciences. The aim of the present study is the first theoretical justification for a key stage of the general concept of lactating cows' health management, proposed in our previous paper (I.M. Mikhailenko, 2014). Herein we suggest an approach to programming cow's life cycle from the first to the last economically reasonable lactation. As a result, the risk of animal culling and unnecessary costs are minimized. The problem is solved for the first time in biological science. Our theory is based on developed dynamic and probabilistic statistical models. At its core, this approach provides a science-based standard of animal feeding for optimized lactation during life cycle. The dynamic models for lifetime annual yields reflect animal age and nutrient intake with diet, and the probabilistic statistical modeling, used to control cows' culling due to ill health and diseases, is the most important feature of the developed approach to life-cycle control. All physiological states, from a heifer to the last lactation, are considered, and all the flows within the dairy herd and possible causes for culling are identified. These mathematical models allow assessing the risk of possible livestock losses, which are minimized due to optimized annual diet. The developed algorithm allows to specify adjustments in annual feeding rations during the cow's life cycle (the feeding strategies for dairy cattle), which ensure optimal reproduction rate, optimal number of possible lactation per cow and optimal annual yields. Thus, the use of a lactating cow is normalized resulting in healthy livestock and maximized profitability of milk production. Since the individual approach to cows' feeding is a substantial reserve for increasing profitability of a dairy herd as a whole, the task of life cycle control is regarded at two levels, for an individual and for the herd on average. For a particular herd, the choice to one of the levels depends on whether there are the means to provide individual health control and dosing concentrated feed and food additives. Practically, the use of proposed mathematical models is mainly limited by lack of long-term (10-12 years) surveys of animal health as depended on the actual diet, since these data are necessary for identification and validation of the algorithms, but an experimental model such as 100-150 cows' dairy farm, equipped with systems for health monitoring and feed composition control, could improve the situation.

Keywords: dairy husbandry, reproduction, concept of lactating cows' health control, health, life cycle, lactation period, control algorithms, mathematical modeling.

An increase in milk production on the best dairy farms in a number of regions of the Russian Federation observed within the last decades which was due to the genetic progress, is not accompanied in general by an increase in cows' productive longevity and improvement in the quality of products obtained. Experts estimate the 65 % of profits in dairy cattle breeding to be determined by the duration of the economic use of cows. In Canada, it is 5 lactations for the whole country, in the US it is 4, in the best breeding centers of the Russian Federation it is 3.8, in many commercial farms it is 2,5 or less, while the estimated biological and economic optimum is 7-8 lactations [1-7].

In Scandinavia, attempts to breed cows producing 100 tons of milk in

10 lactations are made. In our country, under the existing conditions, the extension of the productive use of cows by 6 months is equivalent in the economic effect to increasing the number of dairy herd by 12 %. It means that there is a significant resource to enhance the efficiency of dairy cattle breeding due to increased lifetime productivity of cows [8, 9].

Losses from the shortfall in herd replacements in dairy husbandry occupy one of the first places among all economic losses. In addition, each barren cow causes farm loses of at least 25 % of its milk yield per lactation. To this, the cost of treatment, numerous unsuccessful insemination and losses associated with premature culling of valuable, often young cows, should be added [10].

In our previous study [11], a general description of the problem of reproduction in dairy husbandry was given which is the central problem in this important sector for all countries with developed agriculture. In this regard, the general concept of animal health control was proposed [8]. According to this concept, the purpose of control should not to achieve maximum efficiency as it is widely accepted now, but to maximize the profits earned over the lifetime of an animal. This goal can be achieved following the three inter-related requirements: i) providing the duration of the entire life cycle from first to last genetically and economically feasible lactation; ii) obtaining the conditional maximum profit on each lactation life cycle; iii) individual correction of the conditional maximum profit for the cows and the herd in real time.

This paper discusses the theoretical basis for solving the problem of the life cycle control. At the same time, two approaches are possible depending on the technical and technological basis of a particular farm. In the first case, when such a basis takes into account the health status of each cow, the life cycle of individual animals is managed. This is achieved even without the full-scale personal feeding of an animal without creating special conditions, and can be only due to individual dosage of concentrated fodder, vitamins and premixes. In the second case, when the health assessment is performed selectively, and this figure is determined as the average one for the herd, an additional need to manage the average life cycle of the herd arises.

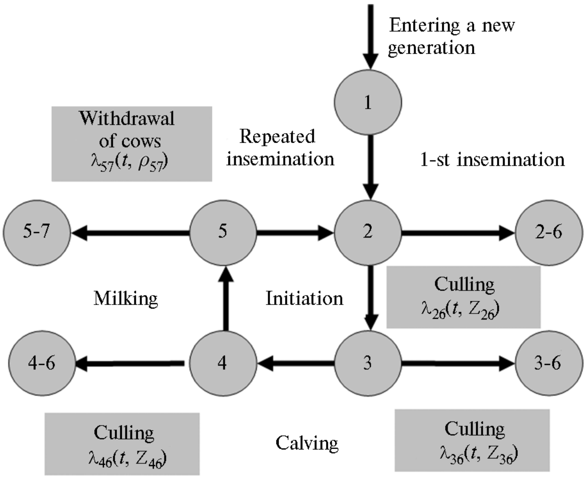
The main condition for the implementation of the full life cycle of a cow is preservation of the reproductive function which can be lost due to inadequate care and feeding and as a result of diseases and disorders. The most common reasons for the culling of lactating cows are metabolic disorders, diseases of digestive system, udder, genitals, postpartum complications, mastitis, joint diseases, and other diseases associated with the peculiarities of feeding and housing of animals. In total there are more than 20 types of diseases resulting in culling of cows and in termination of their life cycle [10].

Since the appearance of any disease is probabilistic in its nature, the onset of any next lactation is regarded as a random event (continued individual life cycle). This results in generation of random flows in the general herd passing from one state to another (Fig.). Of all the possible states, we are interested only in those for which culling is likely, resulting in the termination of the life cycle of an animal and in the changes the structure of the herd.

We introduce the notation of culling flows λ:

from state s = 2 (springer heifers) — $\lambda_{26}(t, Z_{26})$;
from state s = 3 (dry cows) — $\lambda_{36}(t, Z_{36})$;
from state s = 4 (newly calved cows) — $\lambda_{46}(t, Z_{46})$;
from state s = 5 (lactating cows) — $\lambda_{57}(t, \rho_{57})$,

where $\lambda_{ij}(t, Z_{ij})$ is the number of cows culled per unit of time (day), which is generally regarded as the intensity of culling; Z_{ij} is nosological culling vectors which include all types of diseases characteristic of the respective condition leading to culling; ρ_{57} is a rule for the exclusion of lactating cows from circulation.



Graph diagram of livestock turnover in the herd: 1— heifers, 2 — springer heifers, 3 — dry cows, 4 — newly-calved cows, 5 — lactating cows, 6 — culled cows, 7 — cow excluded from turnover based on the efficiency criterion.

If the culling flows in the states of s = 2, 3, 4 is the result of imperfection and inadequacy of the herd state on average control, the formation of the culling flow of lactating cows (s = 7) is the local control component directly focused on obtaining additional profits.

General algorithms of lactating cows' life cycle control. At individual control of lactating cows life cycle, one should keep in mind that of all the states in which the animal can be, s = 5 (lactation) and s = 6 (culling) are most important for the criterial assessment. Each of these states is random and depends on the health of the animal. Therefore, the individual life-cycle control of an individual animal should minimize the risk of losses related to the non-occurrence of the next lactation.

The expression for the risk of the health control in an individual animal is as follows:

$$R(n^*, U_n) = \sum_{n=1}^{n^*-1} [M_n^* - (1-p_n(Z))\cdot(c_n\Pi(U_{1n}) - r(U_{1n}))] - p_n(Z)c_k,$$
$$\Pi_n(U) \le \Pi_n^*;\ p_n \lhd 0,5;\ p_{n=1} = 0, \qquad (1)$$

where n = 1, 2, ..., n^* are the numbers of lactations (from the first to the last economically feasible one); M_n^* is a defined program of profiting from a cow throughout the life cycle; c_n is milk price projections, $\Pi(U_{1n})$ is the annual milk yield throughout the life cycle (function of control vector U_{1n}, the components of which are the expenses of all types of nutrients in the diet); $p_n(Z)$ is a likelihood of culling n-th lactation cows, depending on the disease vector Z; $r(U_{1n})$ is the annual cost of breeding per cow (function of control vector U_{1n}); c_k is the cost of a culled cow; Π_n^* is genetic program of productivity of the appropriate breed.

Expression (1) is the difference between the income at the time of culling ($n = n^*$) and the damage from the animal loss because of culling which is weighted to the state probabilities.

Achieving the goal of the life cycle control in cows corresponds to the minimization of the risk in the number of lactations n^* and the sequence of control vectors over all lactations:

$$R(n^*, U_{1n}) \xrightarrow[n^*,\, U_{1n=1}\dots U_{1n=n^*}]{} \min,$$
$$\Pi_n(U) \le \Pi_n^*,\ p_n \lhd 0,5;\ p_{n=1} = 0. \qquad (2)$$

In the life cycle control in herd on average, the expression for the risk will be as follows:

$$R(n^*,\tilde{U}_n)=\sum_{n=1}^{n^*-1}[\tilde{M}_n^* -(1-\tilde{p}_n(Z_n))\cdot(c_n\tilde{\Pi}_n(\tilde{U}_{1n})-r(\tilde{U}_{1n}))]-c_k\tilde{p}_n(Z_n), \tag{3}$$
$$\tilde{\Pi}_n(U_n)\le\Pi_n^*;\ \tilde{p}_n\lhd 0{,}5;\tilde{p}_{n=1}=0\ ,$$

where $\tilde{p}_n(Z)$ is a likelihood of non-occurrence of the n-th lactation (culling) on average for the herd defined by the disease vector Z for «average for the herd» state of health; $\tilde{\Pi}_n$ is annual milk yield on average for the herd; $\tilde{U}_{1n}$ is the vector of control in the corresponding period of lactation (on average for the herd); $r(\tilde{U}_{1n})$ is the expenses for diet (on average for the herd).

Like for the individual life cycle control, the control on average for the herd corresponds to criterion minimization (1) according to the number of cycles and the sequence of control vectors in individual lactation periods:

$$R(n^*,\tilde{U}_{1n})\xrightarrow[n^*,\tilde{U}_{1n=1}\ldots\tilde{U}_{1n=n^*}]{}\min, \tag{4}$$
$$\tilde{\Pi}_n(\tilde{U}_n)\le\Pi_n^*;\ p_n\lhd 0{,}5;\tilde{p}_{n=1}=0\ .$$

The solution of (2) and (4) is a sequence of control vectors for all lactation periods U_{1n} and $\tilde{U}_{1n}$, as well as the optimal number of lactations n^*.

This will require mathematical models that take into account all the parameters of states for the estimation of the risk of life cycle control. According to the analysis of domestic and foreign publications, nothing like that has been presented yet [12-27]. The main focus of such development is creation of approximating functions maximally reflecting the shape of maximum lactation curves, but they are not suitable for control tasks. New models that reflect age dynamics of cows and culling processes are proposed below.

Models of individual and herd on average lifetime milk yield are:

$$\begin{aligned}&\Pi_{n+1}=a\Pi_n+b^TU_{1n}+c^Tf[n],\\&n\in(1,n^*-1),\ \Pi(1)=\Pi_0;\\&f^T[n]=[n\ n^2\ n^3],\\&U_{1n}=HR_n,\end{aligned} \tag{5}$$

where Π is annual milk yield, kg; U_1 is the vector of nutrient elements spent per year, R is the vector of nutrient components (diet), H is a matrix of nutrient content in fodder ingredients; with $f^T[n]=[n\ n^2\ n^3]$ as the time functions vector taking into account the age of the animal.

Models of individual and herd on average cow culling are:

$$\begin{aligned}&X_n=DU_n,\\&Y_n=X_n-\Delta,\\&C_n=WY_n,\\&z_n=0,\ \text{if}\ \ \rho_z=h_z^TC_n\lhd\delta_z;\\&z_n=z_n,\ \text{if}\ \ \rho_z=(h_z^TC_n)\ge\delta_z\ ;\\&n=n^*,\ \text{if}\ \ z_n\ne 0\ ;\\&p(z_n)=\frac{\rho_z}{\delta_z}\ .\end{aligned} \tag{6}$$

$$p_n(Z)=\sum_{z=1}^{N_z}p(z_n)\ , \tag{7}$$

where X_n is the vector of health status parameters of lactating cows at the n lactation (average value of health status in the interval of lactation $\tilde{X}_n$); Y_n is the vector of diagnostic features equal to the dimension of the vector of health status parameters; Δ is the vector of allowable health values, deviations of which are the diagnostic indicators; C_n is the vector of pathological syndromes which is a linear combination of vectors of diagnostic symptoms; W is a matrix of linear combinations of diagnostic symptoms; h is the vectors of discriminant functions of disease detection resulting in culling; $P\{(h_Z^T C_n \cdot t_\Sigma) \in \Omega_Z\}$ is a probability of discriminant function values hitting in the range of permissible values; t is total time of the health status parameter exceeding allowable values; Ω_Z is the areas of allowable values of disease detection discriminant values.

The algorithm of individual solutions for the animal. The meaning of the task is to minimize the risk

$$R(n^*, U_{1n}) = \sum_{n=1}^{n^*-1} [M_n^* - (1-p_n(Z)) \cdot (c_n \Pi_n(U_{1n}) - r(U_{1n}))] - p_{n^*}(Z) c_k, \quad \Pi_{ni}(U) \le \Pi_n^*; p_n \lhd 0,5;\ p_{n=1} = 0 \tag{8}$$

to the following dynamic system:

$$\Pi_{n+1} = a\Pi_n + b^T U_{1n} + c^T f[n], \quad n \in (1, n^* - 1),\ \Pi(1) = \Pi_0; \quad f^T = [n\ n^2\ n^3], \quad U_{1n} = HR_n. \tag{9}$$

The Hamiltonian of the system (8), (9) is:

$$H_n = (M_n^* - (1-p_n(Z))(c_n \Pi(R_n) - C_R^T R_n) - \frac{1}{n^*} p_n(Z_n) c_k) + +\lambda(a\Pi_n + b^T HR_n + c^T f[n]), \tag{10}$$

with the corresponding conjugate variable model:

$$\lambda_{n+1} = -\frac{\partial H_n}{\partial \Pi} = -a\lambda_n + (1 - p_n(Z) c_n), \tag{11}$$

and the optimization problem is to minimize the criterion (8) by using the Hamiltonian (10) and the conjugate variable (11). A detailed algorithm of this solution is shown below.

In the above problem, the program for profit lactation M_n^*, the price of milk for lactation periods c_n, cows cost c_k, the price of fodder and fodder additives are the initial information which determines the cost of feeding per lactation (U_{1n}) and the parameters of mathematical models of a, b^T, c^T, h^T, δ_Z. In addition, there are «phase» restrictions due to health, defining genetic limitations and the possibility of culling:

$$\Pi(U_{1n}) \le \Pi_n^*; p_n \lhd 0,5;\ p_{n=1} = 0.$$

With recent notes, the optimization problem of the individual life cycle is as follows:

$$R(n^*, R_n) = \sum_{n=1}^{n^*-1} [M_n^* - (1-p_n(Z)) \cdot (c_n \Pi_n(R_n) - r(R_n))] - p_n(Z_n) c_k \xrightarrow[R_1, \ldots R_{n^*} \in \Omega_R]{} \min. \tag{12}$$

Incremental algorithm. Thus, there is the resulting algorithm.

Step 0. Cyclic algorithm variable $i = 0$, initial duration of life cycle, that is the maximum possible number of lactations n_i^*, initial program of «life feeding» as a sequence of the animal diet vectors R_{ni} with $n = 1, 2, \ldots n_i^*$, and the sequence of culling probabilities p_{ni} are set.

Step 1 for i = 1. For the interval of $n = 1, 2, \ldots n^*$ the system is solved:

$$\begin{aligned}
&\Pi_{n+1,i} = a\Pi_{n,i} + b^T U_{1n,i} + c^T f[n],\\
&n \in (1, n_i^* - 1),\ \Pi_{n,i}(1) = \Pi_{0i};\\
&f^T[n] = [n\ n^2\ n^3],\\
&U_{1n,i} = HR_{n,i};\\
&X_{n,i} = DU_{n,i},
\end{aligned} \tag{13}$$

then the sequence of culling probabilities for all lactations is computed

$$\begin{aligned}
&Y_{n,i} = X_{n,i} - \Delta,\\
&C_{n,i} = WY_{n,i},\\
&z_{n,i} = 0,\ \text{if}\ \rho_{zn,i} = (h_z^T C_{n,i}) \lhd \delta_z,\\
&n_{i+1}^* = n_i^*,\ \text{if}\ z_{n,i} = 0,\\
&z_{n,i} = z_n,\ \text{if}\ \rho_{zn,i} = (h_z^T C_{n,i}) \geq \delta_z,\\
&n_{i+1}^* = n_i,\ \text{if}\ z_{n,i} \neq 0,\\
&p(z_{n,i}) = \frac{\rho_{z,i}}{\delta_z},\\
&p_{n,i}(Z) = \sum_{z=1}^{N_z} p_i(z_n).
\end{aligned} \tag{14}$$

Step 2. Equation for the conjugate variable for each lactation is solved in backward time:

$$\begin{aligned}
&\lambda_{n+1,i} = -\frac{\partial H_{n,i}}{\partial \Pi} = -a\lambda_{n,i} + (1 - p_{n,i}(Z)c_n),\\
&n \in (n_i^* + 1, 1);\ \lambda(n_{i+1}^*) = 0,
\end{aligned} \tag{15}$$

and $\lambda_{n,i}$ array is formed.

Step 3 for i = 1. Partial derivatives of the Hamiltonian to vector of diet for each lactation are calculated:

$$g_{n,i} = \frac{\partial H_{n,i}}{\partial R} = (1 - p_{n,i})C_R + \lambda_{n,i} H^T b. \tag{16}$$

Step 4 for i = 1. Sequence of vectors of diet is specified by a lactation:

$$\begin{aligned}
&R_{n,i+1} = R_{n,i} + \Delta_{n,i} g_{n,i},\ \text{if}\ \Pi_{n,i+1} \leq \Pi_n^*,\\
&R_{n,i+1} = R_{n,i},\ \text{if}\ \Pi_{n+1} \rhd \Pi_n^*,
\end{aligned} \tag{17}$$

followed by transition back to Step 1.

Step 5 for i > 1. Partial derivatives of the Hamiltonian to vector of diet for lactations are calculated:

$$\begin{aligned}
&g_{n,i} = \frac{\partial H_{n,i}}{\partial R} = (1 - p_{n,i}(Z))C_R + \lambda_{n,i} H^T b +\\
&+ (c_n \Pi_{n,i} - C_R R_{n,i} - \frac{c}{n_{n,i}^*}) \frac{\partial p_{n,i}(Z)}{\partial R};
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\frac{\partial p_{n,i}(Z)}{\partial R} = \sum_{z=1}^{N_z} [\frac{\partial p_{i,n}(z_n)}{\partial R}] = \sum_{z=1}^{N_z} [\frac{\partial \rho_{z,i}}{\partial R}],\\
&\frac{\partial \rho_{z,i}}{\partial R} = H^T D^T W^T h.
\end{aligned} \tag{19}$$

Step 6 for i > 1. Sequence of diet vectors is specified:

$$R_{n,i+1} = R_{n,i} + \Delta_{n,i} g_{n,i}, \text{ if } \Pi_{n,i+1} \le \Pi_n^*, \text{или } p_n(Z) \le 0,45; \quad (20)$$
$$R_{n,i+1} = R_{n,i}, \text{ if } \Pi_{n+1} \rhd \Pi_n^*, \text{ или } p_n(Z) \rhd 0,45,$$

with transition back to Step 2 if $R^*_{n,i} \ge \delta_p$; if this condition is not satisfied, the process is terminated.

The result of the problem solution is a sequence of optimal annual diets for all lactation periods R^*_n, the limit of lactation number n^*, and optimum annual milk yield Π_n^*, which are used as integral constraints in the management of each lactation period.

The algorithm for the life cycle control averaged for the herd differs from individual algorithm in the state variables only. As a result, in addition to the average herd optimal annual feed rations for all lactation periods R^*_n and optimal average annual milk yields Π_n^* the expected number of retirement cows $N_n^* = p_n(\tilde{Z})N$, with N as total herd livestock at a farm, and the overall risk for dairy cattle herd $\tilde{R}(n^*)$ are obtained. Due to the fact that the article deals with only a general theoretical approach, we do not describe the content of all the vectors of state and control. This can be most clearly shown in practical examples which will be done in next papers.

Thus, the theory of the lactating cows' life cycle control is grounded. In essence, it means minimizing risk of losses from culling and costs of breeding. The theory is based on new dynamic models of annual milk yield and probabilistic statistical models of culling due to cows' diseases and economic feasibility. At its core, the solution of this problem provides science based standards for the animal feed for all lactations of the life cycle ensuring the given rate of reproduction. The problem of the practical implementation of the proposed theory is the lack of information (monitoring) concerning the health status of animals for a prolonged period (at least 10 years) depending on the actual feed rations. This is due to the fact that in the modern farm veterinary services the relevant functions are lost. The situation can be improved only by creating a model dairy farm of 100-150 animals equipped with all necessary systems for monitoring the lactating cows' health and the biochemical composition of feed. Identification of the proposed mathematical models by monitoring information will make it possible to put into practice the lactating cows' life cycle control and solve the reproduction problems in dairy husbandry.

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